On the Dynamical Determination of the Regge Pole Parameters

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The purpose of this article is to outline a method, based on the principles of analyticity and unitarity in the t channel, which may help to determine some dynamical properties of the Regge parameters $\alpha(t)$ and $\beta(t)$.

In the introduction we list various applications of this method, and discuss the role of crossing symmetry and unitarity in all three channels in relation to the uniqueness of the solutions.

We derive the unitarity condition satisfied by the partial wave amplitude $A(l, t)$, for l complex, for a relativistic, two-body scattering process. Upon neglecting intermediate states of more than two particles, the unitarity condition can be expressed in terms of Regge parameters. An approximate form for the unitarity condition, accurate at low t, is next derived. This form will be used for numerical work.

In Section III we show, in the relativistic case, that the functions $\alpha(t)$ and $\beta(t)$, describing a *boson* Regge pole, are analytic with only right-hand cuts, in addition to those arising from the crossing of Regge trajectories. Our proof is based on two assumptions: (1) validity of the Mandelstam representation; (2) analyticity of $A(l, t)$ in the whole l plane, with at most poles and essential singularities. The consequence of the existence of essential singularities at $l = -1, -2, -3, \cdots$ in relation to $\alpha(t)$ and $\beta(t)$ is especially discussed. Finally, we note in this section how the preceding results are modified if the Regge pole being considered is a Fermion.

In Section IV we write dispersion relations for $\alpha(t)$ and $\beta(t)$. These, together with the unitarity condition of Section II, constitute a tentative method for the dynamical determination of the Regge parameters. We outline an extension of our method which is appropriate for discussions of Fermion Regge poles. The behavior of $\alpha(t)$ and $\beta(t)$ at the elastic or inelastic thresholds is derived, and applied to perform subtractions in the dispersion relation for $\beta(t)$.

Finally, in Section V, we turn specifically to π-π scattering and discuss an approximation which might possibly lead to a reasonably accurate description

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of this process. Estimates of the range of validity of the approximation are made.

I. INTRODUCTION

The past year has seen a wealth of new experimental evidence indicating the existence of many new states frequently interpreted as dynamical resonances. At such a time, it is especially important to have at hand a method with which to discuss the properties of these composite states.

The basis for such a method was laid by Regge (*1*), who studied the behavior of the scattering amplitude, as determined by the Schrödinger equation, when dynamical resonances are present. He was able to show, for large momentum transfer t, that these amplitudes take an asymptotic form $\sim\beta(s)t^{\alpha(s)}/\sin\pi\alpha(s)$. A prescription to obtain the Regge pole expansions of two-body field theory amplitudes was soon given (*2*), and the conjecture made that the nucleon (*3*), or possibly even all strongly interacting particles (*4*), were composite in the sense that they would all lie on Regge trajectories and appear in the dispersion relations with the Regge asymptotic behavior.

So far, attention has been directed: (a) towards establishing rigorously various properties of the complex angular momentum theory in potential scattering (and, to a lesser extent, in field theory); (b) to the problem of understanding the relationship of the Regge asymptotic behavior to the question of bound states and subtractions in the Mandelstam representation; and (c) to predict the results of various high-energy experiments and their relation to low-energy resonances in crossed channels. This work has been summarized in reports given at the recent CERN Conference (*5–8*).

We discuss in this paper a possible method[1] to determine some dynamical properties of the Regge pole parameters $\alpha(t)$ and $\beta(t)$. The quantity $\alpha(t)$, it will be recalled, gives the location of a Regge pole in the complex angular momentum plane, while $\beta(t)$ is related, at the pole, to the coupling strength of the Regge particle exchanged.

The dynamical determination of the Regge pole parameters is of interest for several reasons.

(i) The application of the Regge pole hypothesis has been made to discuss, for example, $\pi\pi$, πN, and NN-scattering. Here, two experimental facts stand out: (a) the total cross sections appear to approach a constant value as s gets larger and larger; (b) the angular distributions show characteristic diffraction peaks.

[1] After completion of this work, we received a paper by G. F. Chew (*5*) in which this problem is discussed from a viewpoint similar in spirit but quite different in substance from ours.

From the optical theorem, we know

$$\sigma_{\text{tot}}(s) \xrightarrow[s \to \infty]{} (16\pi/s) \operatorname{Im} A(s, 0). \tag{1}$$

Combined with the Regge asymptotic form for the amplitude, this means (*9*) that $\alpha_i(0) \leqq 1$, for each Regge trajectory i. This suggested (*2*, *10*) the existence of a Regge trajectory with the quantum numbers of the vacuum and $\alpha(0) = 1$, the Pomeranchuk trajectory, which would result in a *constant* total cross section in the high-energy limit. Moreover, exchange of Regge particles was shown to lead, in the simplest cases, to angular distributions with the characteristic diffraction form

$$d\sigma/dt \xrightarrow[s \to \infty]{} F(t)s^{2\alpha(t)-2}. \tag{2}$$

Once the functions $\alpha(t)$ and $\beta(t)$ have been determined by some method, Eq. (2) will become a definite and precise prediction for the angular distribution, rather than a prediction about its general shape.

Secondly, if we are able to determine $\alpha(t)$ and $\beta(t)$ for the Pomeranchuk trajectory, we could predict the position and width of any resonances that might lie on this trajectory (*11*). The same also holds, of course, for all other trajectories.

(ii) Gell-Mann (*7*) has proposed a set of conditions which, if satisfied by a given trajectory, will ensure that such a trajectory will not produce a ghost. Whether or not the Pomeranchuk trajectory, for example, satisfies this condition is a question that can be settled only on the basis of a detailed knowledge of the functions $\alpha_p(t)$ and $\beta_p(t)$.

(iii) The Regge pole conjecture seems to clarify considerably the question of bound states and their relation to the number of subtractions in the Mandelstam representation (*1*, *3*, *12*). One finds, according to the Regge hypothesis, an asymptotic behavior $A(s, t) \to_{s\to\infty} \beta(t)s^{\alpha_m(t)}/\sin \pi\alpha_m(t)$, where $\alpha_m(t)$ is the position of the Regge pole which lies furthest to the right in the complex angular momentum plane. Suppose that for some t, the Re $\alpha_m(t) < 0$. Then the amplitude will converge as $s \to \infty$ at a rate which, according to Froissart (*9*), precludes any arbitrary subtractions in s. By analytic continuation in t, one finds that all subtraction terms are determined. In this sense, the assumed Regge asymptotic behavior for the amplitudes provides a set of boundary conditions which completes the S-matrix description of a two-body scattering process. It is most important to see the above properties of $\alpha_m(t)$ resulting from a dynamical determination of this quantity. We should like to remark, however, that the question of the number of arbitrary quantities needed in the theory (*12*) still appears puzzling to us. This is because it is not yet clear that a dynamical method can be given for calculating $\alpha(t)$ *uniquely*, without arbitrary constants appearing.

Thus, although there may be no undetermined subtraction constants, there may be undetermined constants in $\alpha(t)$. We have not yet investigated this question with regard to our equations for determining $\alpha(t)$ and $\beta(t)$.

In this paper, the principles of analyticity and unitarity are applied to derive dynamical equations for the Regge pole parameters $\alpha(t)$ and $\beta(t)$.

In Section II, we discuss the analytically continued partial wave amplitude and the unitarity condition relating the Regge parameters. We wish to emphasize strongly at this point some of the assumptions to be made here. These are:

(i) Validity of the Mandelstam representation with real singularities only.

(ii) That the partial wave amplitudes may be analytically continued to complex l without encountering natural boundaries. The possibility of essential singularities, in particular (*13*)[2] those at $l = -1, -2, \cdots$, is not excluded here or in the discussion of the analyticity of $\alpha(t)$ and $\beta(t)$.

(iii) We make use of a "chopped off" unitarity condition in which only two-particle intermediate states are kept. We do not thereby limit ourselves to elastic scattering. This approximation we do not regard as essential.

Whether or not a given set of intermediate states can lead to a reasonable description of a scattering process depends on the particular problem considered. We discuss what such an approximation might consist of in considering the application of our method to $\pi\pi$-scattering.

(iv) The unitarity condition which we shall use for later numerical work is approximate in that it holds for $\alpha(t)$ near the real axis, i.e., when $\text{Im}\,\alpha(t)$ is small. We feel that this approximation can be improved upon once a way is found to express the partial wave amplitude entirely in terms of Regge parameters without a background term. For purposes of correlating high-energy data in the direct channel with low-energy resonances in the crossed channel, which is the most natural and important starting point for the application of our dispersion relations, this approximation already seems reasonable.

(v) Recent discussions (*14, 15*) of Fermion Regge poles indicate that the pole parameters are discussed more conveniently as functions of $\sqrt{t}$, rather than t. In this event, the dispersion relations we derive must be modified. This is carried out in Section IV.

(vi) The situation in which Regge poles cross accidentally as indicated in potential theory (*16–18*) is discussed elsewhere (*18*).

In Section III we discuss the analyticity of $\alpha(t)$ and $\beta(t)$ and show that these functions have only right-hand cuts, in addition to any cuts arising from the crossing of two Regge trajectories. Here we discuss a possible essential singularity

[2] This possibility was raised by Gribov and Pomeranchuk in connection with their discussion of the consistency of the conditions of unitarity and the analytic continuation of the partial wave amplitude. We learned of their work first from M. Gell-Mann (private communication), to whom we are also grateful for a helpful discussion on this point.

in the partial wave amplitude at $l = -1, -2, \cdots$, and show that it is consistent with unitarity and does not alter the fact that $\alpha(t)$ and $\beta(t)$ have only the above-mentioned cuts.

In Section IV, analyticity and unitarity are used to write dispersion relations for $\alpha(t)$ and $\beta(t)$. We consider possible subtractions in the dispersion relations, and the threshold behavior of α and β. Other conditions that may aid in determining the solution are discussed, in particular the requirements that $\alpha_i(0) \leqq 1$ and $\operatorname{Im} \alpha_i(t) \geqq 0$ in the physical region may be useful.

Finally, in Section V, we turn to $\pi\pi$-scattering and write dispersion relations with subtractions for the Pomeranchuk and the ρ trajectories. The validity of the approximations involved is briefly discussed for this case.

In closing this section we wish to raise the following points. We have made use of a partial wave expansion in the t-channel, a unitarity condition in the t-channel, and the analytic properties of α and β as functions of t. The only place we have been able to use unitarity in the direct channel s is by requiring $\alpha_i(0) \leqq 1$, and crossing symmetry has entered not at all. Our method for the dynamical determination of the Regge parameters therefore refers almost exclusively to one channel. This fact must mean that much physics is left out in our considerations. In particular, no bootstrap mechanism is included here.

At present, we understand these circumstances to mean that our method for the calculation of α and β, in its present restrictive formulation, cannot yield a *unique* determination of these quantities. This can be seen by considering nonrelativistic scattering where the Regge parameters also satisfy the same dispersion relation and unitarity condition, yet the solution is not unique unless the potential is specified. Our hope is that by supplying additional information, such as the condition $\alpha_i(0) \leqq 1$, the threshold behavior and the value of the $\alpha_i(t)$ and $\beta_i(t)$ at some point where it is known from experiment, we may achieve a unique solution. We note, moreover, that supplying additional information as described above should in no way prevent one from making numerous predictions using this method.

The lack of uniqueness which will arise here is of a type that we expect to be removed once we can make full use of crossing symmetry and unitarity in the s and u channels to carry out self-consistent or "bootstrap" calculations, a possibility that should present itself once a representation of the scattering amplitude solely in terms of Regge parameters is achieved.

Further questions arise. What does this method mean in terms of more familiar concepts? Aside from uniqueness problems, can this method lead to a reasonable understanding of the Regge parameters—and all the physics they summarize—or must one include, to list just one alternative, a background term in order to get sensible results? These questions all remain to be investigated.

II. THE ANALYTICALLY CONTINUED PARTIAL WAVE AMPLITUDE AND THE UNITARITY CONDITION

It is well known that the conventional partial wave amplitude (l a nonnegative integer) for the scattering process $a + b \to c + d$ (Fig. 1), in which the particles have, respectively, masses m_a, m_b, m_c, m_d, can be expressed as

$$
\begin{aligned}
A_{ij}(l, t) &= \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{2q_i q_j} Q_l\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}}{2q_i q_j}\right) D^s_{ij}(t, s) \\
&+ \frac{(-1)^l}{\pi}\int_{u_0}^{\infty} \frac{du}{2q_i q_j} \\
&\qquad Q_l\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)}}{2q_i q_j}\right) D^u_{ij}(u, t),
\end{aligned} \tag{3}
$$

if l is large enough so that the integrals in (3) converge.

In the equation, q_j and q_i are the c.m. momenta of the incoming and outgoing particles in the t-channel, respectively, j is the state $a + b$ and i is the state $c + d$, and $D^s_{ij}(t, s)$, $D^u_{ij}(t, u)$ are the absorptive parts of the scattering amplitude $A_{ij}(s, t, u)$ in the s and u channels. Since $Q_l(z)$ is a meromorphic function of l with poles at the negative integers $l = -1, -2, \cdots$, Eq. (3) provides an analytic continuation of $A(l, t)$, if the integrals on the right side of Eq. (3) converge (*19*). For large z, $Q_l(z) \propto 1/z^{l+1}$, hence the integrals in Eq. (3) converge uniformly in the region Re $l >$ Re α if $D^s_{ij}(t, s)$ and $D^u_{ij}(t, u)$ diverge no faster than s^α and u^α, respectively, for large s and u. The factor $(-1)^l$ for l complex can be defined in various ways; for example, we can define it to be either $e^{i\pi l}$ or $e^{-i\pi l}$. However, we observe that $(-1)^l$ for l an integer takes the value ± 1 according as l is even or odd. We can therefore choose the two independent amplitudes

$$
\begin{aligned}
A^{\pm}_{ij}(l, t) &= \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{2q_i q_j} Q_l\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}}{2q_i q_j}\right) D^s_{ij}(t, s) \\
&\pm \frac{1}{\pi}\int_{u_0}^{\infty} \frac{du}{2q_i q_j} Q_l\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)}}{2q_i q_j}\right) D^u_{ij}(t, u),
\end{aligned} \tag{4}
$$

which correspond to amplitudes with plus or minus signature (*2*).

Making use of the formula (*20*)

$$
Q_l(z) = \frac{\Gamma^2(l + 1)}{2\Gamma(2l + 2)}\left(\frac{z - 1}{2}\right)^{-l-1} F\left(l + 1, l + 1; 2l + 2; \frac{2}{1 - z}\right),
$$

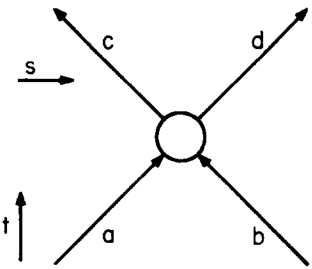

FIG. 1. The scattering process $a + b \to c + d$

we obtain

$$B^{\pm}_{ij}(l, t) \equiv A^{\pm}_{ij}(l, t)/(q_i\, q_j)^l = \frac{\Gamma^2(l+1)}{4\Gamma(2l+2)}\frac{1}{\pi}$$

$$\cdot\left[\int_{s_0}^{\infty}\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2+q_j^2)(m_c^2+q_i^2)} - 2q_i\, q_j}{4}\right)^{-l-1} D^s_{ij}(t, s)\right.$$

$$\times F\left(l+1, l+1; 2l+2,\right.$$

$$\left. -\frac{4q_i\, q_j}{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2+q_j^2)(m_c^2+q_i^2)} - 2q_i\, q_j}\right) ds \tag{5}$$

$$\pm\int_{u_0}^{\infty} du\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2+q_j^2)(m_d^2+q_i^2)} - 2q_i\, q_j}{4}\right)^{-l-1} D^u_{ij}(t, u)$$

$$\times F\left(l+1, l+1; 2l+2;\right.$$

$$\left.\left. -\frac{4q_i\, q_j}{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2+q_j^2)(m_d^2+q_i^2)} - 2q_i\, q_j}\right)\right]$$

valid in the region Re $l >$ Re α. We can observe at this point that the function $B^{\pm}_{ij}(l, t)$ defined in (5) is a real analytic function satisfying

$$B^{\pm}_{ij}(l, t) = (B^{\pm}_{ij}(l^*, t^*))^*. \tag{6}$$

We shall now show that each of the amplitudes $A^{\pm}_{ij}(l, t)$ satisfy the unitarity condition. We first define $(A^{\pm}_{ij}(l, t))^+$ as

$$(A^{\pm}_{ij}(l, t))^+ = \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{2q_i\, q_j} Q_l\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2+q_j^2)(m_c^2+q_i^2)}}{2q_i\, q_j}\right) D^s_{ij}(t^-, s)$$

$$\pm\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du}{2q_i\, q_j} Q_l\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2+q_j^2)(m_d^2+q_i^2)}}{2q_i\, q_j}\right) D^u_{ij}(t^-, u).$$

In the above, $t^- = t - i\epsilon$. For Re $l >$ Re α, we have

$$\frac{[A_{ij}^{\pm}(l, t) - (A_{ij}^{\pm}(l, t))^+]}{2i}$$
$$= \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{2q_i q_j} Q_l\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}}{2q_i q_j}\right)\rho_{ij}(t, s) \qquad (6)$$
$$\pm \frac{1}{\pi}\int_{u_0}^{\infty} \frac{du}{2q_i q_j} Q_l\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)}}{2q_i q_j}\right)\rho_{ij}(t, u).$$

Now, the unitarity condition for the scattering amplitude $A(s, t, u)$ reads (*21*)

$$\rho_{ij}(t, s) = \sum_k \frac{\theta(t - t_k)}{4\omega q_k q_i q_j \pi}\left[\iint \frac{(D_{ki}^s(t, s'))^* D_{kj}^s(t, s'')\, ds'\, ds''}{\sqrt{x^2 + x_k'^2 + x_k''^2 - 2xx_k'x_k'' - 1}} + \iint \frac{(D_{ki}^u(t, u'))^* D_{kj}^u(t, u'')\, du'\, du''}{\sqrt{x^2 + y_k'^2 + y_k''^2 - 2xy_k'y_k'' - 1}}\right], \qquad t > t_0 \qquad (7)$$

In Eq. (7),

$$x = [s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}]/2q_iq_j,$$
$$x_k' = [s' - m_c^2 - m_{k1}^2 + 2\sqrt{(m_c^2 + q_j^2)(m_{k1}^2 + q_k^2)}]/2q_iq_k,$$
$$x_k'' = [s'' - m_a^2 - m_{k1}^2 + 2\sqrt{(m_a^2 + q_j^2)(m_{k1}^2 + q_k^2)}]/2q_jq_k,$$
$$y_k' = [u' - m_c^2 - m_{k2}^2 + 2\sqrt{(m_c^2 + q_i^2)(m_{k2}^2 + q_k^2)}]/2q_iq_k,$$
$$y_k'' = [u'' - m_a^2 - m_{k2}^2 + 2\sqrt{(m_a^2 + q_j^2)(m_{k2}^2 + q_k^2)}]/2q_jq_k,$$

where q_k is the c.m. momentum in the intermediate state k which contains two particles whose masses are m_{k1} and m_{k2}, and ω is one half the c.m. energy of the system. The integration above is over the region where

$$x > x_k'x_k'' + \sqrt{(1 - x_k'^2)(1 - x_k''^2)}$$

for the first term in (7), and a similar region for the second term,

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0, \end{cases}$$

t_k is the threshold for the intermediate state k, and t_0 the mass squared of the multiparticle state with the lowest energy and the same quantum numbers as state i. Likewise,

$$\rho_{ij}(t, u) = \sum_k \frac{1}{4\omega q_k q_i q_j \pi}\left[\iint \frac{(D_{ki}^s(t, s'))^* D_{kj}^u(t, u'')\, ds'\, du''}{\sqrt{x^2 + x_k'^2 + y_k''^2 + 2xx_k'y_k'' - 1}} + \iint \frac{(D_{ki}^u(t, u'))^* D_{kj}^s(t, s'')\, du'\, ds'')}{\sqrt{x^2 + y_k'^2 + x_k''^2 + 2xx_k''y_k' - 1}}\right]\theta(t - t_k), \qquad t > t_0 \qquad (8)$$

with a similar range of integration. From the formula (*22*)

$$\int_0^\infty \frac{Q_l(x)\theta(x - x_1 - x_2 - \sqrt{(1 - x_1^2)(1 - x_2^2)})}{\sqrt{x^2 + x'^2 + x''^2 - 2xx'x'' - 1}}\, dx = Q_l(x_1)Q_l(x_2), \quad (9)$$

we obtain by substituting (7) and (8) into (6) and some algebraic manipulation

$$\frac{[A_{ij}^{\pm}(l, t) - (A_{ij}^{\pm}(l, t)]^{+}}{2i} = \sum_k \frac{q_k}{\omega} (A_{ki}(l, t))^{+} A_{kj}(l, t)\theta(t - t_k), \qquad t > t_0, \quad (10)$$

valid in the region Re $l >$ Re α. Now both sides of Eq. (10) are analytic functions of l, and by analytic continuation (10) holds in the whole region of l where $A(l, t)$ is analytic.

For the case of equal mass, elastic scattering, the argument of Q_l in $A^{\pm}(l, t)$ of (10) is always greater than 1. As $Q_l(z)$ is continuous on the real axis of z if $z > 1$, we have in the elastic scattering case

$$(A^{\pm}(l, t))^{+} = (A^{\pm}(l^*, t))^*,$$

and (10) takes the familiar form

$$[A^{\pm}(l, t) - (A^{\pm}(l^*, t))^*]/2i = (q/\omega)(A(l^*, t))^* A(l, t). \quad (11)$$

Writing $A^{\pm}(l, t) = (\omega/q)(S^{\pm}(l, t) - 1)/2i$, we have also from (11)

$$S^{\pm}(l, t)(S^{\pm}(l^*, t))^* = 1. \quad (12)$$

If we take $l = \alpha(t)$, where $\alpha(t)$ is the location of a Regge pole which occurs in some amplitude $A_{ij}(l, t)$, omitting signature and comparing the residue of both sides of (10) at $l = \alpha$, we obtain

$$\frac{\beta_{ij}}{2i} = \sum_k \frac{q_k}{\omega} (A_{ki}(\alpha, t))^{+} \beta_{ki}(t)\theta(t - t_k). \quad (13)$$

Equating the real part and the imaginary part of (12) would give us two algebraic equations relating α and β_{ij} along the positive real axis $t > t_0$.

Several interesting consequences follow from (13). First, as was pointed out by Gribov and Pomeranchuk (*23*), Eq. (13) requires that $\beta_{ij}(t)$ is factorized and we have relations like (*24*)

$$\beta_{ii}(t) = \beta_{ij}^2(t)/\beta_{jj}(t). \quad (14)$$

This can be proven by considering (13) as a matrix equation

$$\beta(t) = v(t)\beta(t) \quad (15)$$

where

$$v_{ij}(t) = \frac{2iq_j}{\omega} (A_{ji}(\alpha, t))^{+}\theta(t - t_j).$$

Except for some special values of t, $v(t)$ can have only one eigenvector with eigenvalue unity. In this case, if we view $\beta_i(t) = (\beta_{1i}(t), \beta_{2i}(t), \cdots, \beta_{ni}(t))$, $i = 1, 2, \cdots, n$ as a vector, then $\beta_i(t) = v(t)\beta_i(t)$ and (14) follows, for $t > t_0$. We shall prove in the next section that $\beta(t)$ is an analytic function of t; hence by analytic continuation (14) is true for all t.

Secondly, from (14) we see that if any two of $\beta_{ij}(t)$, $\beta_{ii}(t)$, and $\beta_{jj}(t)$ are nonzero, then the third is nonzero. Hence, if a Regge pole occurs in any two of the amplitudes $A_{ii}(l, t)$, $A_{ij}(l, t)$, $A_{jj}(l, t)$, it automatically occurs in the third. The unitarity condition thus implies that the *same* Regge poles occur in all channels (*2*, *3*, *24*). Also, since in the unitarity condition (11), we include only those intermediate states k which have the same conserved quantum numbers as states i and j, a Regge pole trajectory is characterized by a set of conserved quantum numbers. Since $A^{\pm}(l, t)$ satisfy (13) separately, a Regge pole trajectory has a definite signature (*2*).

Equation (13) also leads directly to the form of the unitarity condition as it will be used in this work. If we now write

$$A_{ij}(l, t) \approx \frac{\beta_{ij}(t)}{-\pi(2\alpha(t) + 1)(l - \alpha(t))}, \tag{16}$$

valid for $l \approx \alpha(t)$, then we obtain by substituting (16) into (13)

$$\beta_{ij}(t) = \frac{-1}{\operatorname{Im} \alpha(t)} \sum_k \frac{q_k}{\omega} \frac{\beta_{ki}^*(t)\beta_{kj}(t)}{\pi(2\alpha^*(t) + 1)} \theta(t - t_k). \tag{17}$$

It will prove very convenient to introduce, instead of $\beta_{ij}(t)$, the real analytic function

$$C_{ij}(t) = -\frac{\beta_{ij}(t)e^{-i\pi\alpha(t)}}{\pi(2\alpha(t) + 1)}, \tag{18}$$

which, as follows directly from (17), satisfies the unitarity condition

$$C_{ij}(t)e^{\pi i\alpha^*(t)} = \frac{1}{\operatorname{Im} \alpha(t)} \sum_k \frac{q_k}{\omega} C_{ki}^* C_{kj} \theta(t - t_k). \tag{19}$$

We wish to emphasize here, however, that the unitarity condition as expressed in (17) is approximate in two important respects.

First, multiparticle intermediate states of more than two particles have not been included. The extent to which a scattering process can be described by two-body intermediate states is not clear, here or in any such application of the unitarity condition. We shall discuss this question briefly for $\pi\pi$-scattering in Section V. We note, however, that this short-coming can probably be removed, at least in principle, when techniques for handling multiparticle intermediate states are developed.

Second, we have approximated $A_{ij}(l, t)$ by (16), which is valid only for

$l \approx \alpha(t)$, and hence is true only for Im $\alpha(t)$ small. Attempts to improve this are being made. At any rate, we know that, for some range of t extending upwards from the elastic threshold, Im $\alpha(t)$ is indeed small, and by putting enough subtractions in the dispersion relations we can expect that the contribution from large t is not significant. We might therefore expect that the functions $\alpha(t)$ and $C(t)$ which we obtain are accurately given for t in the above-mentioned range. This point is also discussed further in Section V.

II. THE ANALYTICITY OF $\alpha(t)$ AND $\beta(t)$

In this section we shall present arguments to make plausible the hypothesis that the functions $\alpha(t)$ and $\beta(t)$, for a *boson trajectory*, are analytic functions of t. We shall present two different "proofs": (i) starting from the Mandelstam representation and the assumed Regge behavior (*2*) of the amplitude $A(s, t)$; and (ii) from the assumption that $A(l, t)$ can be analytically continued to the whole l plane. In the proof, the possibility of essential singularities, in particular (*13*) those at $l = -1, -2, \cdots$, is indicated as well as the possibility of the crossing of two Regge trajectories. If there are complex singularities in the two-body scattering amplitude $A(s, t)$, then an extension of our second proof will show that $\alpha(t)$ and $\beta(t)$ may have additional singularities, the location of which will be determined by that of the singularities in $A(s, t)$.

If we assume the Mandelstam representation and the Regge asymptotic behavior to be valid, then we have

$$A_{ij}(s, t) \xrightarrow[|s| \to \infty]{} \frac{\beta(t)}{\sin \pi\alpha(t)} \frac{\Gamma(\alpha(t) + \frac{1}{2})}{\Gamma(\alpha(t) + 1)} \cdot \frac{1}{\sqrt{\pi}} \left(\frac{e^{-i\pi}s}{4q_i\, q_j}\right)^{\alpha(t)} \frac{1}{2}\,(1 \pm e^{-i\pi\alpha(t)}). \tag{20}$$

Since we can vary s arbitrarily, $\alpha(t)$ and $\beta(t)$ must be analytic functions of t with the same cut in t as $A(s, t)$ for fixed s, i.e., a right-hand cut from $t_0 \to \infty$ and a left-hand cut from $\sum m_i^2 - s - u_0$ to $-\infty$, where t_0 and u_0 are the elastic thresholds in the t and u channels. However, the location of the cut in $\alpha(t)$ and $\beta(t)$ must be independent of s, so if we let $s \to \infty$, we see that no left-hand cut can in fact be present. Now the absorptive part in the s channel

$$D^s(s, t) = \frac{A(s + i\epsilon, t) - A(s - i\epsilon, t)}{2i}, \quad s > 0 \tag{21}$$

is real in the physical region of the s-channel, therefore

$$\beta(t)\, \frac{\Gamma(\alpha(t) + \frac{1}{2})}{\Gamma(\alpha(t) + 1)}\, \frac{1}{\sqrt{\pi}} \left(\frac{e^{-i\pi}s}{-4q_i\, q_j}\right)^{\alpha(t)}$$

is real in the physical region of the s-channel. Together with the fact that $\alpha(t)$ and $\beta(t)e^{-i\pi\alpha(t)}$ have no cut on the real axis up to t_0, we see that they are real on the real axis up to t_0. We conclude, therefore, that $\alpha(t)$ and $\beta(t)e^{-i\pi\alpha(t)}$ are both real analytic functions of t, with only a right-hand cut starting from t_0. The above argument is not applicable at a point where two Regge trajectories cross.

It is also possible to establish the desired analytic properties of $\alpha(t)$ and $\beta(t)$ without assuming the Regge asymptotic behavior for the amplitude $A(s, t)$.

We first evaluate explicitly the discontinuity across the left-hand cut of $A(l, t)$. We shall, for simplicity, take $m_a = m_b = m_c = m_d = 1$ and $s_0 = u_0 = 4$. We then obtain

$$\begin{aligned} h^{\pm}(l, t) &= \frac{A^{\pm}(l, t+i\epsilon) - A^{\pm}(l, t-i\epsilon)}{2i} \\ &= -\frac{1}{\pi}\int_4^{-t} \frac{2ds}{t-4} Q_l\left(1 + \frac{2s}{t-4-i\epsilon}\right) \rho_{su}(s, 4-t-s) \\ &\mp \frac{1}{\pi}\int_4^{-t} \frac{2du}{t-4} Q_l\left(1 + \frac{2u}{t-4+i\epsilon}\right) \rho_{su}(4-u-t, u) \\ &+ \frac{1}{2}\int_4^{-(t-4)} \frac{2ds}{t-4} e^{-il\pi} P_l\left(-1 - \frac{2s}{t-4}\right) D^s(t+i\epsilon, s) \\ &\qquad \pm \frac{1}{2}\int_4^{-(t-4)} e^{-il\pi} \frac{2du}{t-4} P_l\left(-1 - \frac{2u}{t-4}\right) D^u(t-i\epsilon, u), \end{aligned} \tag{22}$$

for $t < 0$.

Since the range of integration in Eq. (22) is finite, this equation gives $h^{\pm}(l, t)$ for all l. We thus see from Eq. (22) that $h^{\pm}(l, t)$ is a meromorphic function of l, with simple poles at the negative integers $l = -1, -2, -3, \cdots$. It was shown by Gribov and Pomeranchuk (13) that an essential singularity is required to exist in $A(l, t)$ at these points.

If we write

$$A^{\pm}(l, t) = \frac{t^n}{\pi}\int_{-\infty}^{0} dt \frac{h^{\pm}(l, t')}{(t'-t)t'^n} + A_1^{\pm}(l, t), \tag{23}$$

where n is large enough for the integral to converge, then $A_1^{\pm}(l, t)$ is an analytic function of t with a right-hand cut only. Since no Regge pole can come from the term

$$\frac{t^n}{\pi}\int_{-\infty}^{0} dt' \frac{h^{\pm}(l, t')}{(t'-t)t'^n},$$

all Regge trajectories are determined by

$$F^{\pm}(\alpha, t) \equiv \frac{1}{A_1^{\pm}(\alpha, t)} = 0;$$

or

$$\frac{d\alpha(t)}{dt} = -\frac{\partial F(\alpha, t)}{\partial t} \Bigg/ \frac{\partial F(\alpha, t)}{\partial \alpha}. \tag{24}$$

Here the signature has been omitted.

We shall first investigate the right-hand side of (24) at the points $\alpha = -1, -2, -3, \cdots$. Let us assume that the most singular term of $F(l, t)$ at $l = -n$, is of the form (25) exp $\sum_i a_i(t)/(l+n)^i$, where i is a number (not necessarily an integer). Then we have, at $\alpha = -n$,

$$\frac{d\alpha}{dt} = -\sum \frac{a_i(t)}{(\alpha+n)^i} \Bigg/ -\sum \frac{i a_i(t)}{(\alpha+n)^{i+1}} = 0. \tag{25}$$

This result is unchanged if the most singular term of $F(l, t)$ has the form

$$e^{\sum_i a_i(t)/(l+n)^i}$$

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Thus the right-hand side of (24) is regular at $\alpha = -n$, for this type of essential singularity.

The right-hand side of (24) is thus seen to be an analytic function in t with only a right-hand cut, and entire in α, provided that $\partial F(\alpha, t)/\partial\alpha \neq 0$ for all (α, t). Solving (24) gives α as an analytic function of t with a right-hand cut only. Then $\beta(t)$, which is equal to $-\pi(2\alpha(t) + 1)$ Res $(A_1(l, t))_{l=\alpha(t)}$, is also an analytic function of t with only a right-hand cut.

If, at a given point (α_0, t_b), we have $\partial F(\alpha, t)/\partial\alpha = 0$ besides $F(\alpha, t) = 0$, which is the condition that two or more Regge poles cross at the point, then $\alpha(t)$ is not analytic at t_b, since $d\alpha(t)/dt$ equals infinity there. Assume $\alpha(t_b)$ is not infinite, then t_b is a branch point for $\alpha(t)$ and a branch cut will arise.

Since $A_{ij}(l, t)/(q_i q_j)^l$ is a real analytic function of t and l, $F(l, t)/(q_i q_j)^l$ is a real analytic function of t and l. Thus for t negative and real, $F(\alpha, t) = 0$ implies $F(\alpha^*, t) = 0$. Therefore, either there are two Regge pole trajectories $\alpha_1(t)$ and $\alpha_2(t)$ which are complex conjugate to each other for negative t, and hence satisfy

$$\alpha_1(t) = \alpha_2^*(t^*) \tag{26a}$$

for all t (as $\alpha_1(t)$ and $\alpha_2(t)$ are analytic functions of t), or

$$\alpha(t) = \alpha^*(t) \tag{26b}$$

for t negative and $\alpha(t)$ is a real analytic function of t. In the latter case, $\beta_{ij}(t)/(q_iq_j)^\alpha$ is a real analytic function of t. In the discussion that follows, we shall assume that this in fact is the case.

The location of the right-hand cut of $\alpha(t)$ and $\beta(t)$ is seen to coincide with that of the right-hand cut in t of the scattering amplitude $A(s, t)$, hence it starts from the mass squared of the lowest energy multiparticle state which has the same conserved quantum numbers (except for the angular momentum) as $\alpha(t)$. For example, if we consider the Regge trajectory $\alpha_\omega(t)$ which gives the 3π resonance $I = 0$, $J = 1$ at 787 *Mev*, the branch cut starts at $9m_\pi^2$.

For a fermion Regge trajectory, it has been found (*14*, *15*) that the Regge parameters are best discussed as functions of $\sqrt{t}$, in order to avoid kinematic singularities. Consider the scattering of a fermion by a spinless boson, for example. There are two partial-wave amplitudes, $A(j^+, \sqrt{t})$ and $A(j^-, \sqrt{t})$, corresponding to the two states with the total angular momentum j and parity $(-1)^{j \pm 1/2}$. Both of them are analytic functions of $\sqrt{t}$ with branch cuts: (i) from $\sqrt{t_0}$ to ∞, where t_0 is the energy squared of the lowest mass state having the appropriate quantum numbers; (ii) from $-\sqrt{t_0}$ to $-\infty$; (iii) from $-i\infty$ to $i\infty$. The third kind of branch cut corresponds to the left-hand cut in t in the boson case. A generalization of the arguments presented in this section then shows that $\alpha(\sqrt{t})$ and $\beta_{ij}(\sqrt{t})$ are analytic functions of $\sqrt{t}$ with a right-hand cut from $\sqrt{t_0}$ to ∞ and a left-hand cut from $-\sqrt{t_0}$ to $-\infty$, in addition to those arising from the crossing of two Regge trajectories.

IV. DISPERSION RELATION FOR THE REGGE POLE PARAMETERS

In the preceding section we have shown, assuming the validity of the Mandelstam representation and that $A(l, t)$ is an analytic function of l (possibly with essential singularities), that $\alpha(t)$ and $\beta(t)$ are analytic functions of t with branch cuts only along the positive real axis. Application of Cauchy's theorem to these functions then leads to dispersion relations of the following general form:

$$\alpha(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\alpha(t')}{t' - t}\, dt' \tag{27}$$

$$\beta(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\beta(t')}{t' - t}\, dt'. \tag{28}$$

Here Γ is the contour of a region containing t in which $\alpha(t)$ and $\beta(t)$ are analytic. If $\alpha(t)$ and $\beta(t)$ have only branch cuts whose locations are known, it would generally be more convenient to take the contour Γ along these known cuts.

Suppose, in particular, that $\alpha(t)$ and $\beta(t)$ are associated with a boson trajectory and have the right-hand cuts of Section III. Then we have for each Regge trajectory the *two* algebraic equations (13) which, expressing the unitarity condition, relate $\alpha(t)$ and $\beta(t)$ of all Regge trajectories on the cut along the positive real axis, plus the *two* coupled nonlinear integral equations (27) and (28) which must be satisfied by the *four* unknown functions Re $\alpha(t)$, Im $\alpha(t)$, Re $\beta(t)$, and Im $\beta(t)$ for each Regge trajectory.

For a *Fermion* Regge trajectory which does not cross any other trajectories, the dispersion relation reads (*14*)

$$\alpha(\sqrt{t}) = \frac{1}{\pi}\int_{\sqrt{t_0}}^{\infty} \frac{\operatorname{Im}\alpha_+(t')d\sqrt{t'}}{\sqrt{t'}-\sqrt{t}} + \frac{1}{\pi}\int_{\sqrt{t_0}}^{\infty} \frac{\operatorname{Im}\alpha_-(t')d\sqrt{t'}}{\sqrt{t'}+\sqrt{t}}, \tag{27a}$$

where $\alpha_+(t')$ and $\alpha_-(t')$ are the Regge poles of $A(j^+, \sqrt{t})$ and $A(j^-, \sqrt{t})$ respectively, and similarly for $\beta(\sqrt{t})$. The functions $A(j^+, \sqrt{t})$ and $A(j^-, \sqrt{t})$ satisfy the unitarity condition separately and, as before, it relates $\alpha_\pm(t)$ to $\beta_\pm(t)$ on the branch cuts.

In order to avoid complications arising from the crossing of Regge trajectories, dispersion relations should be written for a set of algebraic functions of $\alpha(t)$ and $\beta(t)$, instead of $\alpha(t)$ and $\beta(t)$ themselves. This is elaborated in another paper (*18*).

We have not yet shown that solutions to the four coupled equations for each trajectory (27, (28), and (13) exist, nor have we investigated under what conditions they may be unique. In the remainder of this section we shall discuss various questions that bear either on the question of subtractions, or on conditions that can be added to help select a unique solution.

The singularity of $\alpha(t)$ and $\beta(t)$ at infinity is not known in the relativistic case. In the case of potential scattering in the potential

$$\int_{m^2}^{\infty} \sigma(\mu^2)\,\frac{e^{-\mu r}}{r}\,d\mu^2,$$

it has been found that (*26*)

$$\alpha_n(q^2) \xrightarrow[|q^2|\to\infty]{} -n - i\,\frac{\int_{m^2}^{\infty}\sigma(\mu^2)d\mu^2}{2q} \tag{29}$$

$$\beta_n(q^2) \xrightarrow[|q^2|\to\infty]{} -\frac{-\pi(2n-1)\int_{m^2}^{\infty} d\mu^2\sigma(\mu^2)}{2q^2}, n = 1, 2, 3, \cdots, \tag{30}$$

where $\alpha_n(q^2)$ is the nth Regge trajectory and q^2 is the energy. We see from this that *no* subtractions are *necessary*, although it may be convenient for practical purposes to make some. In the relativistic case, therefore, it may be reasonable to conjecture that $\alpha(t)$ and $\beta(t)$ have no singularity at infinity. This is the most appealing conjecture from the theoretical point of view. However, to be on the safe side, we shall usually prefer to make subtractions. How many subtractions are to be made really depends on the specific problem and on what one is willing to supply from the outside as subtraction constants as compared to what one wishes to predict.

We next wish to investigate the behavior of $\alpha(t)$ and $\beta(t)$ near a threshold. This is of interest for two reasons: (i) the behavior of $\alpha(t)$ and $\beta(t)$ near a threshold can be rigorously established. This information on the functional form of of $\alpha(t)$ and $\beta(t)$, used as a boundary condition in connection with Eqs. (27), (28), and (19), should help to make the solution of these equations unique. (ii) If subtractions in $\beta(t)$ are made for those values of t corresponding to thresholds, the subtraction constants can be shown to vanish, hence no additional parameters are introduced.

The behavior of $\alpha(t)$ near the threshold in the relativistic many-channel problem has been shown by Barut (*27*) to be the same as in the elastic scattering case, if only two-particle intermediate states are considered. We shall apply his arguments to obtain the threshold behavior for $\beta(t)$.

The unitarity condition for $B(l, t)$ takes the matrix form

$$\frac{B(l, t + i\epsilon) - B^*(l^*, t + i\epsilon)}{2i} = B^*(l^*, t + i\epsilon)\rho(t)B(l, t + i\epsilon) \tag{31}$$

where $\rho(t)$ is the matrix with elements

$$\rho_{ij}(t) = \delta_{ij}\, \theta(t - t_i)(q_i q_j)^{l+1/2}/\omega. \tag{32}$$

Making use of the fact that $B(l, t) = B^*(l^*, t^*)$, we have

$$\frac{B^{-1}(l, t + i\epsilon) - B^{-1}(l, t - i\epsilon)}{2i} = -\rho(t), \tag{33}$$

which gives the discontinuity of $B(l, t)$ across the right-hand cut. Write

$$B^{-1}(l, t) = (Y(l, t) + R)/\cos \pi l \tag{34}$$

where

$$R_{ij}(l, t) = q_i^{2l+1}\, e^{-i\pi(l+1/2)}\delta_{ij}\, \theta(t - t_i)/\omega.$$

Then $Y(l, t)$ is analytic in t with only a left-hand cut, since R has the same discontinuity across the right-hand cut as $B^{-1}(l, t)$. The matrix $Y(l, t)$ is the analogue of the Y function previously introduced in the nonrelativistic case (*20*, *28*). From (34) we have

$$B(l, t) = \frac{\operatorname{adj}(Y + R)}{\det(Y + R)} \cos \pi l, \tag{35}$$

and the Regge poles are given by the zeroes of $\det(Y + R)$. Now suppose t is near the threshold t_i of state i so that q_i is small, then we have

$$\begin{aligned}\det(Y(\alpha, t) + R(\alpha, t)) \\ = \det(Y(\alpha, t) + R'(\alpha, t)) + R_{ii}(\alpha, t)[Y(\alpha, t) + R(\alpha, t)]_{ii} = 0,\end{aligned} \tag{36}$$

where $R'(\alpha, t)$ is the matrix obtained from R by deleting the element $R_{ii}(\alpha, t)$, and $[Y(\alpha, t) + R(\alpha, t)]_{ii}$ is the cofactor of the element ii of $(Y + R)$. If we write $F(\alpha, t) = \det(Y(\alpha, t) + R'(\alpha, t))/[Y(\alpha, t) + R(\alpha, t)]_{ii}$, Eq. (36) becomes

$$F(\alpha, t) = -q_i^{2\alpha+1} e^{-i\pi(\alpha+1/2)}/\omega, \tag{37}$$

where $F(\alpha, t)$ is an analytic function of t in the neighborhood of the threshold t_i.

If $\operatorname{Re} \alpha(t_i) > -\frac{1}{2}$, then $F(\alpha(t_i), t_i) = 0$. Expanding $F(\alpha, t)$ in a Taylor series and writing $\alpha_i = \alpha(t_i)$, we find

$$\left.\frac{\partial F(\alpha, t)}{\partial \alpha}\right|_{\substack{\alpha=\alpha_i \\ t=t_i}} (\alpha - \alpha_i) + \left.\frac{\partial F(\alpha, t)}{\partial t}\right|_{\substack{\alpha=\alpha_i \\ t=t_i}} (t - t_i) = -q_i^{2\alpha_i+1} e^{-i\pi(\alpha_i+1/2)}/\omega,$$

or

$$\alpha(t) \approx \alpha_i + a(t - t_i) + b q_i^{2\alpha_i+1} e^{-i\pi(\alpha_i+1/2)}. \tag{38}$$

Also

$$\begin{aligned}\beta_{ij}(t) &= -(q_i q_j)^{\alpha(t)} \pi(2\alpha(t) + 1) \operatorname{Res} (B(l, t))_{l=\alpha(t)} \\ &\approx C(q_i q_j)^{\alpha_i}\end{aligned}$$

If $\operatorname{Re} \alpha_i < -\frac{1}{2}$, then (37) shows $F(\alpha_i, t_i) = \infty$. Writing (37) in the form

$$\frac{1}{F(\alpha, t)} = -\omega q_i^{-2\alpha(t)-1} e^{i\pi(\alpha+1/2)}$$

and expanding $1/F(\alpha, t)$ in a Taylor series now gives us

$$\alpha(t) \approx \alpha_i + e(t - t_i) + f q_i^{-2\alpha_i-1} e^{i\pi(\alpha_i+1/2)}, \tag{39}$$

and

$$\beta(t) \approx g q_i^{-3\alpha_i-2} q_j^{\alpha_i}, \tag{40}$$

where a, b, c, d, e, f, g are constants. At the elastic threshold these constants are all real and are related (*20, 28*). We see that for inelastic two-body scattering, $\beta_{ij}(t)$ will vanish at *two* points, i.e., will vanish if $q_i \to 0$ *or* if $q_j \to 0$, provided that either $\operatorname{Re} \alpha(t_i) > 0$ or $\operatorname{Re} \alpha(t_i) < -\frac{2}{3}$. The function $\beta_{ii}(t)$, however, vanishes only at one point. We also see that subtractions made at any threshold of $\beta(t)$ do not introduce new parameters because $\beta_{ij}(t) \to 0$ as $t \to t_i$.

There are two other conditions that we would like to mention here. First, Froissart (*9*) showed that unitarity in the direct channel implies that, for each Regge trajectory i, $\alpha_i(0) \leqq 1$. This condition is the only place where unitarity in the direct channel is used in our work. It should be employed to help select unique solutions of the integral equations. Another condition that might be imposed is $\operatorname{Im} \alpha_i(t) \geqq 0$, if $\operatorname{Re} \alpha_i > -\frac{1}{2}$, and t is physical. This has so far been proven only in potential theory (*1*)—therefore it would be preferable to have it appear as a consequence of our equations.

V. A METHOD OF APPROXIMATION FOR THE CALCULATION OF REGGE PARAMETERS

We have seen in the preceding sections that the analyticity of α and β together with the unitarity condition give us a set of coupled integral equations satisfied by the Regge parameters of all trajectories. In practice, however, it is not feasible to solve the integral equations which involve the Regge parameters of *all* trajectories. We shall see in this section that an approximation to the unitarity condition would decouple $\alpha(t)$ and $\beta_{11}(t)$ from all the other Regge parameters. (Here, the subscript 1 indicates the lowest mass state with the same quantum numbers characterizing the Regge trajectory $\alpha(t)$.) The approximation of the unitarity condition is good only at low energy (in the crossed channel), and as a result, the Regge parameters obtained from this approximation can be hoped to be accurate only at low energy.

We shall illustrate our method by applying it, in the case of $\pi\pi$-scattering, to the Regge parameters of the Pomeranchuk trajectory, which is characterized by the quantum numbers of the vacuum and $\alpha_p(0) = 1$, and to the ρ trajectory which gives rise to the 2π resonance ($I = 1$, $J = 1$) at 750 Mev. Both of these trajectories have a branch cut starting from $4m_\pi^2$, (hereafter, we take the mass of a pion to be unity), as there are 2π states which have the same quantum number as the vacuum, or the quantum numbers of the ρ meson. In the elastic region ($4 < t < 16$) the approximate unitarity condition (17) reads

$$1 = -\frac{1}{\operatorname{Im} \alpha(t)} \sqrt{\frac{t-4}{t}} \frac{\beta_{11}(t)}{\pi(2\alpha(t)+1)}, \tag{41}$$

for both the P trajectory and the ρ trajectory. The subscript "1" now denotes the 2π state. We see that in this approximation, the unitarity condition for each trajectory involves only the two parameters $\beta_{11}(t)$ and $\alpha(t)$. The Pomeranchuk and ρ trajectories are decoupled.

In terms of the function C_{ij}, introduced in Eq. (18), the unitarity condition (41) takes the form

$$\frac{C_{11}(t)e^{i\pi\alpha(t)}}{\operatorname{Im} \alpha(t)} = \sqrt{\frac{t}{t-4}}, \tag{42}$$

accurate for $4 < t < 16$ and Im α small. For the ρ trajectory, we crudely estimate Im $\alpha_\rho(29) < 0.1$, which indicates that approximating $A(\alpha^*, t)$ by the Regge pole term is reasonably accurate perhaps up to $t \sim 30$. The inelastic region starts at 16. However, if we approximate the 4π states by a $\pi\omega$ state or a 2ρ state, we see that (42) would not be modified until $t \sim 45$ or $t \sim 119$. As to the $K\bar{K}$ state, the threshold is at $t \sim 51$. This, together with the fact that β_{ij} is still quite small even slightly above threshold of state i or j, as it vanishes at the threshold if Re $\alpha > 0$ or Re $\alpha < -2/3$, indicates that (42) is probably a moderately good approximation up to $t \sim 30$ or more.

As Eq. (42) is valid only for small t, we should make as many subtractions in the dispersion relations as possible *without introducing unknown quantities*. This should reduce the large t contribution to the dispersion integrals. If $C_{ij}(t)$ vanishes at threshold, we can make two subtractions if $i \neq j$, and one if $i = j$. Now $\alpha_p(0) = 1$, hence $\alpha_p(4) > 1$. Also, previous work (*29*) indicates that it is quite safe to assume $\alpha_\rho(4) > 0$. Therefore, $C_p(4) = C_\rho(4) = 0$, where the subscript "11" has been dropped. We thus have

$$\mathrm{Re}\, \alpha_p(t) = 1 + \frac{t}{\pi} P \int_4^\infty \frac{\mathrm{Im}\, \alpha_p(t')\, dt'}{t'(t'-t)}, \tag{43a}$$

$$\mathrm{Re}\, C_p(t) = \frac{(t-4)}{\pi} P \int_4^\infty \frac{\mathrm{Im}\, C_p(t')\, dt'}{(t'-4)(t'-t)}, \tag{43b}$$

$$\mathrm{Re}\, \alpha_\rho(t) = 1 + \frac{t-29}{\pi} P \int_4^\infty \frac{\mathrm{Im}\, \alpha_\rho(t')\, dt'}{(t'-29)(t'-t)}, \tag{43c}$$

$$\mathrm{Re}\, C_\rho(t) = \frac{(t-4)}{\pi} P \int_4^\infty \frac{\mathrm{Im}\, C_\rho(t')\, dt'}{(t'-4)(t'-t)}. \tag{43d}$$

Subtractions have been made for C_ρ and C_p at the threshold $t = 4$, for α_p at $t = 0$, and for α_ρ at $t = 29$. The unitarity condition (42) will be used for $t > 4$. By equating real and imaginary parts of (42), we obtain

$$\frac{C_\rho(t) e^{i\pi\alpha_\rho(t)}}{\mathrm{Im}\, \alpha_\rho(t)} = \sqrt{\frac{t}{t-4}}, \tag{44a}$$

$$\mathrm{Re}\, C_\rho(t) \sin \pi\alpha_\rho(t) + \mathrm{Im}\, C_\rho(t) \cos \pi\alpha_\rho(t) = 0, \tag{44b}$$

$$\frac{C_p(t) e^{i\pi\alpha_p(t)}}{\mathrm{Im}\, \alpha_p(t)} = \sqrt{\frac{t}{t-4}}, \tag{44c}$$

and

$$\mathrm{Re}\, C_p(t) \sin \pi\alpha_p(t) + \mathrm{Im}\, C_p(t) \cos \pi\alpha_p(t) = 0. \tag{44d}$$

The eight equations (44) and (43) can be solved for the eight parameters Re $C_p(t)$, Im $C_p(t)$, Re $\alpha_p(t)$, Im $\alpha_p(t)$, Re $C_\rho(t)$, Im $C_\rho(t)$, Re $\alpha_\rho(t)$, Im $\alpha_\rho(t)$.

Numerical results will be reported in a forthcoming paper. It should be noted that Eqs. (44) may be moderately accurate only up to about $t = 30$. Thus the Regge parameters obtained may be accurate only for $t \sim 30$.

An improvement to (44) by taking care of some inelastic states seems feasible. In view of the considerable improvement obtained (*30*) by including the $\pi\omega$ state in the self-consistent calculation of the mass and the coupling strength of the ρ meson, we may want to include the scattering processes $\pi + \omega \rightarrow \pi + \omega$ and $\pi + \omega \rightarrow \pi + \pi$. Equation (44) will then involve the C functions for these two processes. This will make the numerical calculation more complicated, but adds no more difficulties in principle.

As has been stressed in the introduction, the results of a calculation such as this will allow comparison with experiment at many points. In particular[3] Im $\alpha_\rho(29)$, $\alpha_\rho'(29)$, and $C_\rho(29)$ will determine the width and coupling constant of the $I = 1$, $J = 1$ 2π resonance. The calculation will indicate whether a $J = 2$ resonance on the Pomeranchuk trajectory exists at ~ 1 *Bev* (if its range of validity can be extended up to $t \sim 50$), if so its position and width will be predicted; $C_p(0)$ is related to the total $\pi\pi$ cross section; $\alpha_\rho(0)$ is related to $\sigma_{\pi^+p} - \sigma_{\pi^-p}$ and α_p, α_ρ, C_p, and C_ρ are all related to measurable angular distributions for $t \lesssim 0$.

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